

Indirect control of quantum system via accessor: pure coherent control without system excitation

H. C. Fu^{1‡}, Hui Dong², X. F. Liu³, C. P. Sun^{2§}

¹ School of Physics Science and Technology, Shenzhen University, Shenzhen 518060, P. R. China

² Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100080, P. R. China

³ Department of mathematics, Peking University, Beijing 100871, P. R. China

Abstract. A pure indirect control of quantum systems via quantum accessor is investigated. In this control scheme, we do not apply any external classical excitation fields on the controlled system and we control a quantum system via a quantum accessor and classical control fields control the accessor only. Complete controllability is investigated for arbitrary finite dimensional quantum systems and exemplified by 2 and 3 dimensional systems. The scheme exhibits some advantages; it uses less qubits in accessor and does not depend on the energy-level structure of the controlled system.

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‡ E-mail: hcfu@szu.edu.cn

§ E-mail: suncp@itp.ac.cn

1. Introduction

Quantum control is a coherence-preserving manipulation of a quantum system, which enables a time evolution from an arbitrary initial state to an arbitrary target state [1-4]. It was first proposed by Huang *et. al.* [5] in 1983 and was mainly used to control chemical reaction in its early days [6]. Recently it has attracted much attention due to its connection to quantum information processing. Actually the universality of quantum logic gates can be understood from viewpoint of complete controllability in quantum control [7]. Conventional quantum control is the coherent control of quantum systems using classical external fields. Controllability of this *semi-classical* control is well studied [8], especially the complete controllability of finite dimensional quantum systems using Lie algebra method [9, 10] graph method [11] and transfer graph method [12]. Lie algebra approach plays important role in the investigation in both the classical control [13] and the quantum control.

In some circumstances in quantum information processing, there is need to control qubits using quantum controllers such as quantum accessor and environment. For example, in connection with the fundamental limit of quantum information processing and influence of decoherence to quantum control, we have proposed an indirect scheme for quantum control where the controller is also quantum [14]. To avoid switching the couplings between qubits, Zhou *et. al.* introduced the so-called encoded qubits to realize the universal quantum computation with local manipulation of physical qubits only [15]. Here the physical qubits do not involve the quantum computation and play the role of quantum controllers. Recently Hodges *et. al.* proposed an universal indirect control of nuclear spins using a single electron spin acting as an accessor driven by microwave irradiation of resolved anisotropic hyperfine [16], which has important application for spin based solid state quantum information processing. Therefore the control of quantum systems using quantum controllers has significant application in quantum information processing and has attracted much attention recently. Authors of this paper proposed the conception of the *indirect control* of quantum systems where the quantum systems are controlled via a quantum accessor and the classical control fields control the accessor only [17]. Similar works were proposed in different context [20] for spin-1/2 particles. Romano [18] and Pechen [19] considered the incoherent control induced by environment modeled as quantum radiation fields.

In our previous paper [17], we proposed a scheme for the control of an arbitrary finite dimensional quantum system using a quantum accessor modeled as a qubit chain with XY-type neighborhood coupling. We find the conditions of way of coupling between the controlled system and accessor and the minimal length of qubit chain to ensure the complete control of the controlled system. However, besides the classical control fields controlling on the accessor, we also apply a constant classical field on the controlled system to excite the system through dipole interaction. Without the excitation field, the system is not completely controllable and for the 2-dimensional case, underlying Lie algebra is the Symplectic algebra $sp(4)$, rather than $su(4)$. Another disadvantage of this scheme is that the controllability depends on the structure of energy levels of the controlled system. In this paper we shall remove the excitation field and propose a *pure* indirect control scheme where the external control fields control the accessor *only*. We shall see that, in comparison with the scheme in [17], the new scheme proposed in this paper exhibits some advantages besides the removal of excitation field, for example, it uses less qubits of the accessor for complete control of the controlled system and there is no particular requirements

on the structure of the energy levels of controlled system.

The remaining part of this paper is organized as follows: we formulate the control system without system excitation in Sec. 2 and then introduce the *selection* operators and apply it to the study of controllability of two energy level system in Sec. 3. The case of 3-dimensional system is investigated in Sec. 4. The general approach of controllability of indirect control of arbitrary finite-dimensional systems is investigated in Sec. 5. We conclude in Sec. 6.

2. Indirect control system

In this section we shall formulate the indirect control system and fix the notations we will use later on. Suppose that the system to be controlled is an N -dimensional quantum system described by the following Hamiltonian

$$H_S = \sum_{i=1}^N E_i e_{ii} = \sum_{i=1}^{N-1} \epsilon_i h_i, \quad (1)$$

where E_i 's are eigen energy of the system, e_{ij} is an $N \times N$ matrix with matrix elements $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$, $h_i = e_{ii} - e_{i+1,i+1}$ are Cartan generators of the Lie algebra $\mathfrak{su}(N)$ and $\epsilon_i \equiv E_1 + E_2 + \dots + E_i$. Here we have assumed that $\text{tr} H_S = 0$ without losing generality.

Note that we do not apply any external classical excitation field on the system S as we did in [17]. The excitation field, although it is a constant field, makes the indirect control in [17] not really pure indirect.

The quantum accessor is modeled as a qubit chain with XY-type neighborhood coupling

$$\begin{aligned} H_A &= H_A^0 + H_A^I, \\ H_A^0 &= \sum_{i=0}^M \hbar \omega_i \sigma_z^i, \quad H_A^I = \sum_{i=1}^{M-1} c_i \sigma_x^i \sigma_x^{i+1}, \end{aligned} \quad (2)$$

where $c_i \neq 0$ and

$$\sigma_\alpha^i = 1 \otimes \dots \otimes 1 \otimes \sigma_\alpha \otimes 1 \otimes \dots \otimes 1, \quad (3)$$

namely the σ_α on the site i and 1 on any other sites.

The system and the accessor are coupled as

$$H_I = \sum_{\{\alpha_i\}} \left[\sum_{j=1}^{N-1} \sum_{k=0,\pm 1} g_{(\alpha_i)}^{j(k)} s_j^k \right] \otimes \sigma_{\alpha_1}^1 \dots \sigma_{\alpha_M}^M, \quad (4)$$

where $\{\alpha_i\} = \{\alpha_1, \alpha_2, \dots, \alpha_M\}$ and each $\alpha_i = x, y, z$ rather than just x, y as in previous paper [17], s_j^k is defined as

$$s_j^k = \begin{cases} x_j & \text{when } k = 1; \\ h_j & \text{when } k = 0; \\ y_j & \text{when } k = -1, \end{cases} \quad (5)$$

and

$$\begin{aligned} x_j &= e_{j,j+1} + e_{j+1,j}, \\ y_j &= i(e_{j,j+1} - e_{j+1,j}), \end{aligned} \quad (6)$$

along with h_j constitute the Chevalley basis of the Lie algebra $\mathfrak{su}(N)$ [21].

It is known that when we remove the excitation field, the indirect system is not completely controllable if the $\alpha_i = x, y$ only [17]. In the case of indirect control of 2-level system, the Lie algebra is $\mathfrak{sp}(4)$ with dimension 10, rather than the $\mathfrak{su}(4)$ [17]. However, as the example we presented in [17], we can rotate the system to remove the excitation field, but as price paid the interaction Hamiltonian includes σ_z for the accessor part. So this is why we includes $\alpha_i = z$ in the coupling Hamiltonian (4).

We suppose that we can control each qubit of the accessor completely through external classical fields. The complete control of each qubit in a qubit chain can be achieved via global manipulation [23, 24]. Therefore the total control system is

$$H = H_0 + \sum_{j=1}^M [f_j(t)\sigma_x^j + f'_j(t)\sigma_y^j],$$

$$H_0 = H_S + H_A + H_I, \quad (7)$$

where $f_j(t)$ and $f'_j(t)$ are two independent classical control fields.

In the rest of this paper we shall investigate the complete controllability of the indirect control scheme (7), namely in what conditions the system is completely controllable. More precisely, in what conditions the Lie algebra generated by the skew-Hermitian operators iH_0 , $i1 \otimes \sigma_x^j$ and $i1 \otimes \sigma_y^j$ ($j = 1, 2, \dots, M$) is $\mathfrak{su}(2^M N)$, or its dimension is $(2^M N)^2 - 1$.

3. Selection operators and Indirect control of single qubit

To prove the complete controllability, we define the so-called *selection operators* S_{xy}^k and S_{yx}^k acting on the Pauli's operators of k -th qubit of the accessor

$$S_{xy}^k = \frac{1}{4} \text{ad}_{i\sigma_x^k} \text{ad}_{i\sigma_y^k}, \quad S_{yx}^k = \frac{1}{4} \text{ad}_{i\sigma_y^k} \text{ad}_{i\sigma_x^k} \quad (8)$$

where $\text{ad}_{i\sigma_x^k}$ is the adjoint representation of the Lie algebra $\mathfrak{su}(2)$ of the k -th qubit

$$\text{ad}_X(Y) = [X, Y], \quad \forall X, Y \in \mathfrak{su}(2). \quad (9)$$

From definition (9), it follows

$$S_{xy}^k(*) = \frac{1}{4} [i\sigma_x^k, [i\sigma_y^k, *]],$$

$$S_{yx}^k(*) = \frac{1}{4} [i\sigma_y^k, [i\sigma_x^k, *]]. \quad (10)$$

It is easy to prove that

$$S_{xy}^k(i\sigma_\alpha^k) = \begin{cases} i\sigma_y^k, & \alpha = x; \\ 0, & \alpha = y, z; \end{cases} \quad (11)$$

$$S_{yx}^k(i\sigma_\alpha^k) = \begin{cases} i\sigma_x^k, & \alpha = y; \\ 0, & \alpha = x, z; \end{cases} \quad (12)$$

namely, S_{xy}^k transforms the $i\sigma_x^k$ to $i\sigma_y^k$, S_{yx}^k transforms the $i\sigma_y^k$ to $i\sigma_x^k$ and they annihilate any others. Or in other words, S_{xy}^k can *select* the σ_x^k and change it to σ_y^k from any linear combination of Pauli's matrices.

Now we show how to use those operators in the investigation of complete controllability with the 2-dimensional system as an example. Here both the system

and accessor are single qubit. The Hamiltonian of the system and accessor is as follows

$$\begin{aligned}
H_0 = & \hbar\omega_S\sigma_z \otimes 1 + \hbar\omega_A(1 \otimes \sigma_z) + \\
& + (g_{xx}\sigma_x + g_{yx}\sigma_y + g_{zx}\sigma_z) \otimes \sigma_x \\
& + (g_{xy}\sigma_x + g_{yy}\sigma_y + g_{zy}\sigma_z) \otimes \sigma_y \\
& + (g_{xz}\sigma_x + g_{yz}\sigma_y + g_{zz}\sigma_z) \otimes \sigma_z.
\end{aligned} \tag{13}$$

We suppose we can control the accessor fully

$$H_C = f_1(t)1 \otimes \sigma_x + f_2(t)1 \otimes \sigma_y, \tag{14}$$

where $f_1(t)$ and $f_2(t)$ are two independent classical control fields to control the accessor. The Lie algebra generators are iH_0 , $i1 \otimes \sigma_x$ and $i1 \otimes \sigma_y$ and the generated Lie algebra is denoted by \mathcal{L} . It is obvious that

$$-2^{-1}[i1 \otimes \sigma_x, i1 \otimes \sigma_y] = i1 \otimes \sigma_z \in \mathcal{L}. \tag{15}$$

So we can subtract the second term in H_0 and obtain the Lie algebra element $iH'_0 \equiv iH_0 - i\hbar\omega_A(1 \otimes \sigma_z) \in \mathcal{L}$.

Now we apply the selection operators on the element iH'_0 , yielding

$$S_{xy}(iH'_0) = i(g_{xx}\sigma_x + g_{yx}\sigma_y + g_{zx}\sigma_z) \otimes \sigma_y \in \mathcal{L}, \tag{16}$$

$$S_{yx}(iH'_0) = i(g_{xy}\sigma_x + g_{yy}\sigma_y + g_{zy}\sigma_z) \otimes \sigma_x \in \mathcal{L}. \tag{17}$$

In fact, by evaluating the commutation relation of (16,17) with the generators σ_x and σ_y of accessor in (16) and (17) can be changed to any σ_α ($\alpha = x, y, z$).

We further subtract the terms (16) and (17) from iH'_0 and then calculate its commutation relation with $i1 \otimes \sigma_y$. We find

$$i(g_{xz}\sigma_x + g_{yz}\sigma_y + g_{zz}\sigma_z) \otimes \sigma_\alpha \in \mathcal{L}. \tag{18}$$

If the following condition

$$\det \begin{pmatrix} g_{xx} & g_{yx} & g_{zx} \\ g_{xy} & g_{yy} & g_{zy} \\ g_{xz} & g_{yz} & g_{zz} \end{pmatrix} \neq 0 \tag{19}$$

is satisfied, we find nine Lie algebra elements

$$i\sigma_\alpha \otimes \sigma_\beta \in \mathcal{L}, \tag{20}$$

where $\alpha, \beta = x, y, z$. The condition (19) can be achieved by choosing, for example, $g_{xx} = g_{yy} = g_{zz} = 1$ and any others zero. As we already have $i1 \otimes \sigma_\alpha \in \mathcal{L}$, so we only need to prove $i\sigma_\alpha \otimes 1 \in \mathcal{L}$. For this purpose, we evaluate

$$-2^{-1}[i\sigma_x \otimes \sigma_x, i\sigma_y \otimes \sigma_x] = i\sigma_z \otimes 1 \in \mathcal{L}, \tag{21}$$

$$2^{-1}[i\sigma_x \otimes \sigma_x, i\sigma_z \otimes \sigma_x] = i\sigma_y \otimes 1 \in \mathcal{L}, \tag{22}$$

$$2^{-1}[i\sigma_y \otimes 1, i\sigma_z \otimes 1] = i\sigma_x \otimes 1 \in \mathcal{L}. \tag{23}$$

In summary, the generated Lie algebra has fifteen generators $i\sigma_\alpha \otimes \sigma_\beta$ where $\alpha, \beta = x, y, z, 0$ ($\sigma_0 \equiv 1$) and α, β cannot be 0 simultaneously, and they generate the Lie algebra $\mathfrak{su}(4)$. Therefore the single qubit system is completely controllable under the condition (19).

4. Control of 3-dimensional system

In this section we turn to the indirect control of 3-dimensional quantum system. The Hamiltonian takes the following form

$$\begin{aligned}
H_0 &= H_S + H_A + H_{SA} \\
H_S &= \sum_{i=1}^3 E_i e_{ii} = E_1 h_1 + (E_1 + E_2) h_2 \\
H_A &= \hbar \omega_1 \sigma_z^1 + \hbar \omega_2 \sigma_z^2 + c \sigma_x^1 \sigma_x^2 \\
H_{SA} &= \sum_{\alpha, \beta=x, y, z} \left(g_{\alpha\beta}^{1(0)} h_1 + g_{\alpha\beta}^{2(0)} h_2 + g_{\alpha\beta}^{1(1)} x_1 + g_{\alpha\beta}^{2(1)} x_2 \right. \\
&\quad \left. + g_{\alpha\beta}^{1(-1)} y_1 + g_{\alpha\beta}^{2(-1)} y_2 \right) \otimes \sigma_\alpha^1 \sigma_\beta^2
\end{aligned}$$

where $h_1 = e_{11} - e_{22}$ and $h_2 = e_{22} - e_{33}$ are Cartan elements of Lie algebra $\mathfrak{su}(3)$, and $x_i = e_{i, i+1} + e_{i+1, i}$ and $y_i = i(e_{i, i+1} - e_{i+1, i})$ ($i = 1, 2$) are Chevelley basis of $\mathfrak{su}(3)$ corresponding positive and negative simple roots, respectively. The complete control system is

$$H = H_0 + \sum_{k=1}^2 (f_k(t) 1 \otimes \sigma_\alpha^k + f'_k(t) 1 \otimes \sigma_\alpha^k),$$

where $f_k(t)$ and $f'_k(t)$ are classical control fields.

It is easy to see that $i1 \otimes \sigma_z^k \in \mathcal{L}$. So we can subtract the free Hamiltonian of the accessor from H_0 and obtain the following Lie algebra element

$$H'_0 = H_0 - (\hbar \omega_1 \sigma_z^1 + \hbar \omega_2 \sigma_z^2) \in \mathcal{L}. \quad (24)$$

It is easy to check that

$$\begin{aligned}
S_{yx}^2 S_{yx}^1 (iH'_0) &= i \left(g_{yy}^{1(0)} h_1 + g_{yy}^{2(0)} h_2 + g_{yy}^{1(1)} x_1 \right. \\
&\quad \left. + g_{yy}^{2(1)} x_2 + g_{yy}^{1(-1)} y_1 + g_{yy}^{2(-1)} y_2 \right) \otimes \sigma_x^1 \sigma_x^2 \in \mathcal{L}, \quad (25)
\end{aligned}$$

$$\begin{aligned}
S_{xy}^2 S_{yx}^1 (iH'_0) &= i \left(g_{yx}^{1(0)} h_1 + g_{yx}^{2(0)} h_2 + g_{yx}^{1(1)} x_1 + \right. \\
&\quad \left. g_{yx}^{2(1)} x_2 + g_{yx}^{1(-1)} y_1 + g_{yx}^{2(-1)} y_2 \right) \otimes \sigma_x^1 \sigma_y^2 \in \mathcal{L}, \quad (26)
\end{aligned}$$

$$\begin{aligned}
(1 - S_{xy}^2 - S_{yx}^2) S_{yx}^1 (iH'_0) &= i \left(g_{yz}^{1(0)} h_1 \right. \\
&\quad \left. + g_{yz}^{2(0)} h_2 + g_{yz}^{1(1)} x_1 + g_{yz}^{2(1)} x_2 + g_{yz}^{1(-1)} y_1 + \right. \\
&\quad \left. g_{yz}^{2(-1)} y_2 \right) \otimes \sigma_x^1 \sigma_z^2 \in \mathcal{L}. \quad (27)
\end{aligned}$$

After proper commutation with the external interaction Hamiltonian, we can change the accessor part in Eqs.(25-27) to $\sigma_y^1 \sigma_y^2$, $\sigma_y^1 \sigma_x^2$ and $\sigma_y^1 \sigma_z^2$, respectively. Then we subtract those Lie algebra elements from iH'_0 and obtain the following Lie algebra element

$$\begin{aligned}
iH''_0 &= c \sigma_x^1 \sigma_x^2 + \sum_{\alpha=x, z} \sum_{\beta=x, y, z} \left(g_{\alpha\beta}^{1(0)} h_1 + g_{\alpha\beta}^{2(0)} h_2 + g_{\alpha\beta}^{1(1)} x_1 \right. \\
&\quad \left. + g_{\alpha\beta}^{2(1)} x_2 + g_{\alpha\beta}^{1(-1)} y_1 + g_{\alpha\beta}^{2(-1)} y_2 \right) \otimes \sigma_\alpha^1 \sigma_\beta^2. \quad (28)
\end{aligned}$$

To remove the term $c\sigma_x^1\sigma_x^2$ from iH_0'' , we evaluate the commutation relation between iH_0'' and $i(1 \otimes \sigma_x^1)$, yielding

$$\begin{aligned} iH_0''' &= -2^{-1} [iH_0'', i1 \otimes \sigma_x^1] \\ &= i \sum_{\beta=x,y,z} \left(g_{z\beta}^{1(0)} h_1 + g_{z\beta}^{2(0)} h_2 + g_{z\beta}^{1(1)} x_1 + g_{z\beta}^{2(1)} x_2 \right. \\ &\quad \left. + g_{z\beta}^{1(-1)} y_1 + g_{z\beta}^{2(-1)} y_2 \right) \otimes \sigma_y^1 \sigma_\beta^2 \in \mathcal{L}. \end{aligned} \quad (29)$$

Then we can use the same trick as in (25-27) to prove

$$\begin{aligned} S_{yx}^2(iH''') &= i \left(g_{zy}^{1(0)} h_1 + g_{zy}^{2(0)} h_2 + g_{zy}^{1(1)} x_1 + g_{zy}^{2(1)} x_2 \right. \\ &\quad \left. + g_{zy}^{1(-1)} y_1 + g_{zy}^{2(-1)} y_2 \right) \otimes \sigma_x^1 \sigma_x^2 \in \mathcal{L}, \end{aligned} \quad (30)$$

$$\begin{aligned} S_{xy}^2(iH''') &= i \left(g_{zx}^{1(0)} h_1 + g_{zx}^{2(0)} h_2 + g_{zx}^{1(1)} x_1 + g_{zx}^{2(1)} x_2 \right. \\ &\quad \left. + g_{zx}^{1(-1)} y_1 + g_{zx}^{2(-1)} y_2 \right) \otimes \sigma_x^1 \sigma_y^2 \in \mathcal{L}, \end{aligned} \quad (31)$$

$$\begin{aligned} (1 - S_{xy}^2 - S_{yx}^2)(iH''') &= i \left(g_{zz}^{1(0)} h_1 + g_{zz}^{2(0)} h_2 + \right. \\ &\quad \left. g_{zz}^{1(1)} x_1 + g_{zz}^{2(1)} x_2 + g_{zz}^{1(-1)} y_1 + g_{zz}^{2(-1)} y_2 \right) \\ &\quad \otimes \sigma_x^1 \sigma_z^2 \in \mathcal{L}. \end{aligned} \quad (32)$$

Now we have found six independent Lie algebra elements Eqs.(25-27) and Eqs.(30-32), in which the accessor part can be changed to the same $\sigma_\alpha^1 \sigma_\beta^2$ ($\alpha, \beta = x, y, z$) by evaluating proper commutation with the external interaction Hamiltonian. If the coefficients satisfy the following condition

$$\det \begin{pmatrix} g_{yx}^{1(0)} & g_{yx}^{2(0)} & g_{yx}^{1(1)} & g_{yx}^{2(1)} & g_{yx}^{1(-1)} & g_{yx}^{2(-1)} \\ g_{yy}^{1(0)} & g_{yy}^{2(0)} & g_{yy}^{1(1)} & g_{yy}^{2(1)} & g_{yy}^{1(-1)} & g_{yy}^{2(-1)} \\ g_{yz}^{1(0)} & g_{yz}^{2(0)} & g_{yz}^{1(1)} & g_{yz}^{2(1)} & g_{yz}^{1(-1)} & g_{yz}^{2(-1)} \\ g_{zx}^{1(0)} & g_{zx}^{2(0)} & g_{zx}^{1(1)} & g_{zx}^{2(1)} & g_{zx}^{1(-1)} & g_{zx}^{2(-1)} \\ g_{zy}^{1(0)} & g_{zy}^{2(0)} & g_{zy}^{1(1)} & g_{zy}^{2(1)} & g_{zy}^{1(-1)} & g_{zy}^{2(-1)} \\ g_{zz}^{1(0)} & g_{zz}^{2(0)} & g_{zz}^{1(1)} & g_{zz}^{2(1)} & g_{zz}^{1(-1)} & g_{zz}^{2(-1)} \end{pmatrix} \neq 0, \quad (33)$$

we have that all the elements

$$h_k \otimes \sigma_\alpha^1 \sigma_\beta^2 \in \mathcal{L}, \quad x_k \otimes \sigma_\alpha^1 \sigma_\beta^2 \in \mathcal{L}, \quad y_k \otimes \sigma_\alpha^1 \sigma_\beta^2 \in \mathcal{L}, \quad (34)$$

where $k = 1, 2$ and $\alpha, \beta = x, y, z$.

So we need further to prove $1 \otimes \sigma_\alpha^1 \sigma_\beta^2 \in \mathcal{L}$ and $h_k \otimes 1 \in \mathcal{L}$, $x_k \otimes 1 \in \mathcal{L}$, $y_k \otimes 1 \in \mathcal{L}$. To this end let us evaluate

$$-\frac{1}{2} [ih_k \otimes \sigma_\alpha^1 \sigma_\beta^2, ix_k \otimes \sigma_\alpha^1 \sigma_\beta^2] = y_k \otimes 1 \in \mathcal{L}, \quad (35)$$

$$\frac{1}{2} [ih_k \otimes \sigma_\alpha^1 \sigma_\beta^2, y_k \otimes \sigma_\alpha^1 \sigma_\beta^2] = ix_k \otimes 1 \in \mathcal{L}, \quad (36)$$

$$\frac{1}{2} [ix_k \otimes \sigma_\alpha^1 \sigma_\beta^2, y_k \otimes \sigma_\alpha^1 \sigma_\beta^2] = ih_k \otimes 1 \in \mathcal{L}. \quad (37)$$

In this 3-dimensional system case, we choose all coefficients $g_{x\alpha}^{j(k)} = 0$. Then from iH_0'' we subtract the Lie algebra elements (25-27, 30-32) with the same accessor part $\sigma_\alpha^1 \sigma_\beta^2$ ($\alpha, \beta = x, y, z$) and find

$$i1 \otimes \sigma_x^1 \sigma_x^2 \in \mathcal{L} \quad (38)$$

from which we have $i1 \otimes \sigma_\alpha^1 \sigma_\beta^2 \in \mathcal{L}$. So we have proved the complete controllability of 3-dimensional quantum systems.

Here we would like to note that the Lie algebra $\mathfrak{su}(3)$ has 6 Chevalley basis and therefore we need 6 equations to decouple the terms in Hamiltonian. However, there are nine elements of type $i\sigma_\alpha^1 \sigma_\beta^2$.

5. Complete controllability of finite dimensional quantum system

With experience built in previous two sections, we shall generally investigate the complete controllability of arbitrary finite dimensional quantum systems in this section. In the interaction Hamiltonian H_I there are 3^M coupling terms

$$\left[\sum_{j=1}^{N-1} \sum_k g_{\{\alpha_i\}}^{j(k)} s_j^k \right] \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M. \quad (39)$$

Here we call them *nomial* for convenience. Notice that the accessor part in each nomial is labeled by an index set $\{\alpha_i\}$. In the forthcoming part of this paper we use symbol $\{\alpha_i|n\}$ to denote this index set, in which the number of σ_z in the nomials is not less than n . It is obvious for $\{\alpha_i|n\}$, there are

$$\binom{M}{n} 2^{M-n} \quad (40)$$

nomials in which there are n σ_z 's. One can easily check that the sum of those numbers gives rise to 3^M using binomial formula, the total coupling terms in H_I , as we expected.

In the following we shall first prove each nomials (39) is in Lie algebra \mathcal{L} and then prove the generated Lie algebra is $\mathfrak{su}(N2^M)$.

5.1. Decoupling iH_I to nomials

We first prove that each terms in H_I is in the Lie algebra \mathcal{L} . We shall prove this recursively according to the number of σ_z in each nomial.

We first notice that the element $iH_A^0 \in \mathcal{L}$, so the element $iH^{(0)} \equiv iH_0 - iH_A^0 \in \mathcal{L}$. Without losing generality, we suppose $M \geq 3$ hereafter.

As the first step, we would like to *select* the terms without σ_z in the qubit chain. For this purpose, we first annihilate iH_A^I and the nomials with σ_z 's in iH_I from iH^0 by evaluating the commutation relations

$$\begin{aligned} iH^{(0)1} &\equiv \left[i\sigma_z^M, \left[i\sigma_z^{M-1}, \dots, \left[i\sigma_z^1, iH^{(0)} \right] \dots \right] \right] \\ &= i2^M \sum_{\llbracket \alpha_i \rrbracket} (-1)^{\Delta_{\llbracket \alpha_i \rrbracket}} \left[\sum_{j=1}^{N-1} \sum_k g_{\llbracket \alpha_i \rrbracket}^{j(k)} s_j^k \right] \otimes \\ &\quad \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M \in \mathcal{L}, \end{aligned} \quad (41)$$

where we have used the symbol $\llbracket \alpha_i \rrbracket$ to denote the index set with each $\alpha_i = x, y$ only, and $\Delta_{\llbracket \alpha_i \rrbracket}$ is the number of x in $\llbracket \alpha_i \rrbracket$. Note that the terms iH_A^I is also annihilated as each terms in it has only two neighborhood qubits.

As each index in (41) is either x or y , we can use selection operators to pick up each nomial in Eq.(41)

$$S_{\beta_M, \bar{\beta}_M}^M S_{\beta_{M-1}, \bar{\beta}_{M-1}}^{M-1} \cdots S_{\beta_1, \bar{\beta}_1}^1 (iH^{(0)1}) =$$

$$i \left[\sum_{j=1}^{N-1} \sum_k g_{\{\beta_i\}}^{j(k)} s_j^k \right] \otimes \sigma_{\beta_1}^1 \sigma_{\beta_2}^2 \cdots \sigma_{\beta_M}^M \in \mathcal{L}, \quad (42)$$

where $\beta_i, \bar{\beta}_i = x, y$ and

$$\bar{\beta}_i = \begin{cases} x, & \text{if } \beta_i = y; \\ y, & \text{if } \beta_i = x. \end{cases} \quad (43)$$

So the Eq.(42) implies that we have 2^M Lie algebra elements in which the β_i is either x or y .

As the second step, we further deprive nomials with just one σ_z in qubit chain. We first evaluate the proper commutation with external interaction Hamiltonian to change the $\sigma_{\beta_k}^k$ to $\sigma_{\bar{\beta}_k}^k$ in Eq.(42) and then subtract them from $iH^{(0)}$. We obtain the following Lie algebra element

$$\begin{aligned} iH^{(1)} &\equiv iH_S + iH_A^I \\ &+ i \sum_{\{\alpha_i|1\}} \left[\sum_{j=1}^{N-1} \sum_k g_{\{\alpha_i|1\}}^{j(k)} s_j^k \right] \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M \in \mathcal{L}. \end{aligned} \quad (44)$$

Without losing generality, we consider the case where $\alpha_1 = z$. As in step 1, we would like to annihilate the term iH_A^I and all nomials that has σ_z in sites other than site 1. For this purpose, we evaluate commutation relation on the site 1 with $i\sigma_x^1$ and other sites with σ_z^j . Those M operations change σ_z^1 to σ_y^1 and other σ_x^n to σ_y^n or vice versa for $2 \leq n \leq M$. We have

$$\begin{aligned} iH^{(1)1} &\equiv \left[i\sigma_z^M, \dots, \left[i\sigma_z^2, \left[i\sigma_x^1, iH^{(1)} \right] \dots \right] \right] \\ &= i2^M \sum_{\substack{\{\alpha_i|1\} \\ \alpha_1=z}} (-1)^{\Delta_{\{\alpha_i|1\}}} \left[\sum_{j=1}^{N-1} \sum_k g_{\{\alpha_i|1\}}^{j(k)} s_j^k \right] \otimes \\ &\quad \sigma_y^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M \in \mathcal{L}. \end{aligned} \quad (45)$$

where $\alpha_i = x$ or y for $i = 2, 3, \dots, M$. As each site of the accessor is either σ_x or σ_y , we can use selection operators to find 2^{M-1} elements of \mathcal{L}

$$i \left[\sum_{j=1}^{N-1} \sum_k g_{\{z, \beta_2, \dots, \beta_M|1\}}^{j(k)} s_j^k \right] \otimes \sigma_y^1 \sigma_{\beta_2}^2 \cdots \sigma_{\beta_M}^M \in \mathcal{L}. \quad (46)$$

In fact, we can use the same method to prove that nomials with only one $\alpha_k = z$ on the site k are elements of Lie algebra \mathcal{L} . In total we have $M2^{M-1}$ such type Lie algebra elements.

We can now subtract the element

$$i \left[\sum_{j=1}^{N-1} \sum_k g_{\{z, \beta_2, \dots, \beta_M|1\}}^{j(k)} s_j^k \right] \otimes \sigma_z^1 \sigma_{\beta_2}^2 \cdots \sigma_{\beta_M}^M \in \mathcal{L} \quad (47)$$

which can be obtained from the commutation relation of σ_x^1 with (46), and obtain an element of \mathcal{L} which takes the same form of (44) but there are at least two z in $\llbracket \alpha_i \rrbracket$

$$\begin{aligned} iH^{(2)} &\equiv iH_S + iH_A^I + \\ &i \sum_{\{\alpha_i|2\}} \left[\sum_{j=1}^{N-1} \sum_k g_{\{\alpha_i|2\}}^{j(k)} s_j^k \right] \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M \\ &\in \mathcal{L}. \end{aligned} \quad (48)$$

Suppose that $\alpha_m = \alpha_n = z$ ($m \neq n$). Then we can evaluate the commutation relation of $iH^{(2)}$ with $i\sigma_x^m$, $i\sigma_x^n$ and $i\sigma_z^k$ ($k \neq m, n$), one can easily prove that the element with two z in the Lie algebra \mathcal{L} .

Following the procedure recursively on the number of z in $\{\alpha_i\}$, we can prove all the elements

$$i \left[\sum_{j=1}^{N-1} \sum_k g_{\{\alpha_i\}}^{j(k)} s_j^k \right] \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M \in \mathcal{L}, \quad (49)$$

where each $\alpha_i = x, y, z$. There are 3^M such elements.

As each nomial of the type (49) is a linear combination of $3(N-1)$ elements x_i , y_i and h_i ($i = 1, 2, \dots, N-1$), so we require that the number of qubits is big enough such that

$$3^M \geq 3(N-1), \quad (50)$$

and then choose $3(N-1)$ elements of type (49). Then we further require the determinant of the coefficient matrix

$$\det \left(g_{\{\alpha_i\}}^{i(k)} \right) \neq 0, \quad (51)$$

we find that the elements

$$ix_i \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M, \quad (52)$$

$$iy_i \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M, \quad (53)$$

$$ih_i \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M, \quad (54)$$

are Lie algebra elements. Namely, all nomials in interaction Hamiltonian iH_I are decoupled and each term is in the Lie algebra \mathcal{L} .

5.2. System operators as Lie algebra elements

It is easy to see that

$$\frac{1}{2} [ix_i \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M, y_i \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M] = ih_i \otimes 1_A \in \mathcal{L}, \quad (55)$$

$$-\frac{1}{2} [ih_i \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M, ix_i \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M] = y_i \otimes 1_A \in \mathcal{L}, \quad (56)$$

$$-\frac{1}{2} [ih_i \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M, iy_i \otimes \sigma_{\alpha_1}^1 \sigma_{\alpha_2}^2 \cdots \sigma_{\alpha_M}^M] = iy_i \otimes 1_A \in \mathcal{L}. \quad (57)$$

From those Chevalley basis elements corresponding to simple roots of Lie algebra $\mathfrak{su}(N)$, we can further construct the standard Cartan basis of $\mathfrak{su}(N)$ corresponding any other positive and negative roots. We have in total $N^2 - 1$ such basis elements of \mathcal{L} .

5.3. Accessor elements

Above discussions mean that the Hamiltonian iH_S^0 is an element of \mathcal{L} . So subtracting this element along with iH_I and iH_A^0 , we find that $iH_A^I \in \mathcal{L}$.

It is easy to see that

$$[[iH_A^I, i1 \otimes \sigma_y^1], i1 \otimes \sigma_y^1] = -i4c_1(1_S \otimes \sigma_x^1 \sigma_x^2) \in \mathcal{L} \quad (58)$$

thanks to the condition $c_1 \neq 0$. We further have that

$$\begin{aligned} & [[iH_A^I - ic_1 1_S \otimes \sigma_x^1 \sigma_x^2, i1 \otimes \sigma_y^2], i1 \otimes \sigma_y^2] \\ & = -i4c_2(1_S \otimes \sigma_x^2 \sigma_x^3) \in \mathcal{L} \end{aligned} \quad (59)$$

since $c_2 \neq 0$. Repeating this process we can prove that

$$i(1_S \otimes \sigma_x^j \sigma_x^{j+1}) \in \mathcal{L}, \quad j = 1, 2, \dots, M-1. \quad (60)$$

Then from the Lemma 2 in Ref.[17], we find that

$$i(1_S \otimes \sigma_{[\alpha]}) \in \mathcal{L}, \quad [\alpha] \neq (0, 0, \dots, 0) \quad (61)$$

The number of those type of elements is $4^M - 1$.

5.4. Complete controllability

So far we have proved that if the conditions (50) and (51) are satisfied, the following elements are in Lie algebra \mathcal{L}

$$\begin{aligned} & h_i \otimes 1_A \in \mathcal{L}, \quad ix_i \otimes 1_A \in \mathcal{L}, \quad iy_i \otimes 1_A \in \mathcal{L}, \\ & h_i \otimes \sigma_{[\alpha]} \in \mathcal{L}, \quad ix_i \otimes \sigma_{[\alpha]} \in \mathcal{L}, \quad iy_i \otimes \sigma_{[\alpha]} \in \mathcal{L}, \\ & i(1_S \otimes \sigma_{[\alpha]}) \in \mathcal{L}, \end{aligned} \quad (62)$$

and their corresponding Cartan basis elements. The total number of those Lie algebra elements is

$$(N^2 - 1) + (N^2 - 1)(4^M - 1) + (4^M - 1) = (2^M N)^2 - 1, \quad (63)$$

which is the dimension of Lie algebra $\mathfrak{su}(2^M N)$. This proves the complete controllability of the indirect control system (7).

6. Conclusion

In this paper we have proposed a scheme for the indirect control of finite dimensional quantum systems via quantum accessor modeled as a qubit chain with XY-type coupling. The main results of this paper are as follows:

- Different from our previous paper [17], we do not need to apply an excitation classical field on the controlled system. So this scheme is a *pure* indirect control in the sense that the classical control fields control the accessor only.
- The minimal number M for the complete control of the controlled system is determined by condition (50), while in previous scheme [17], the minimal M is determined by $2^M \geq 2(N-1)$. It is obvious that the scheme proposed here requires less qubits in accessor in comparison to the proposal [17].
- We also notice that in the process of decoupling the interaction Hamiltonian (see Sec.5.1) we do not put any requirements on the structure of energy-level of the controlled system, while in [17], the indirect controllability reduces to the semi-classical control investigated in [9, 10] which depends on the energy-level structure of the controlled system.
- From Eq.(51) we find that the controllability is determined by the way of coupling of controlled system and accessor. So in a practical control protocol we can design a simplest coupling of the controlled system and accessor to ensure the complete control of the controlled system, according to the condition (51).

Therefore we believe the scheme in this paper has wider applicability. As further works we would like to study the concrete control protocol of the indirect control, and examine the graph connectivity for assessing the controllability of quantum systems, as well as applications in quantum information processing.

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